Due Sun

1.4 – Inverses; Algebraic Properties of Matrices

Theorem 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid for matrices A, B, and C and scalars a, b, and c.

a) $A + B = B + A$
(commutative law for matrix addition)

b) $A + (B + C) = (A + B) + C = A + B + C$
(associative law for matrix addition)

c) $(AB)C = A(BC) = ABC$
(associative law for matrix multiplication)

d) $A(B + C) = AB + AC$ (left distributive law)

e) $(B + C)A = BA + CA$ (right distributive law)

f) $A(B - C) = AB - AC$

g) $(B - C)A = BA - CA$

h) $a(B + C) = aB + aC$

i) $a(B - C) = aB - aC$

j) $(a + b)C = aC + bC$

k) $(a - b)C = aC - bC$

l) $a(bC) = (ab)C$

m) $a(BC) = (aB)C = B(aC)$

Pf: (h) For $a(B+C)$ to be equal to $aB+aC$, they must have the same size and equal entries.
By assumption, B & C are the same size.
Scalar mult. doesn't change the size of a matrix, so aB & aC are also the same size as B & C.

$(a(B+C))_{ij}$ is the entry in the ith row and jth column of the matrix on the LHS.

$$(a(B+C))_{ij} = a((B+C))_{ij} = a(b_{ij}+c_{ij})$$
$$= ab_{ij} + ac_{ij} \quad \text{dist. prop of real \#s}$$
$$= (aB)_{ij} + (aC)_{ij}$$

Since corresponding entries are equal, $a(B+C) = aB + aC$.

In general, $AB \neq BA$. In the special cases where $AB = BA$, we say that A and B **commute**.

A **zero matrix**, denoted O, is a matrix whose entries are all zero.

Theorem 1.4.2 Properties of Zero Matrices
If c is a scalar, and if the sizes of the matrices A and O are such that the operations can be performed, then:

a) $A + O = O + A = A$
b) $A - O = A$
c) $A - A = A + (-A) = O$
d) $OA = O$
e) If $cA = O$, then $c = 0$ or $A = O$

However, if $AC = BC$, it does not always follow that $A = B$. Also, we can have $AB = O$ even if $A \neq O$ and $B \neq O$.

$$a(B+C) = a\left(\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}\right)$$

$$= a\left(\begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & \cdots & b_{1n}+c_{1n} \\ b_{21}+b_{21} & b_{22}+c_{22} & \cdots & b_{2n}+c_{2n} \\ \vdots & & & \\ b_{m1}+c_{m1} & b_{m2}+c_{m2} & \cdots & b_{mn}+c_{mn} \end{bmatrix}\right)$$

$$= \begin{bmatrix} a(b_{11}+c_{11}) & \cdots & a(b_{1n}+c_{1n}) \\ & \text{and so on} & \end{bmatrix}$$

$$= \begin{bmatrix} ab_{11}+ac_{11} & \cdots & ab_{1n}+ac_{1n} \\ \vdots & \ddots & \vdots \end{bmatrix}$$

$$= \begin{bmatrix} ab_{11} & \cdots & ab_{1n} \\ \vdots & & \vdots \end{bmatrix} + \begin{bmatrix} ac_{11} & \cdots & ac_{1n} \\ \vdots & & \vdots \end{bmatrix} = aB + aC$$

Let $\overset{\text{matrix}}{A} = \overset{\text{Matrix}}{[a_{ij}]}$ be an $m \times k$ matrix $\quad \overset{\text{entry}}{(A)_{ij}} = \overset{\text{entry}}{a_{ij}}$

and $B = [b_{ij}]$ be a $k \times n$ matrix

$$\underset{m\times k \;\; k\times n}{(A\,B)_{ij}} = \sum_{r=1}^{k} a_{ir} b_{rj}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & & \vdots & & \vdots \\ b_{k1} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix}$$

$$(AB)_{ij} = a_{i\underline{1}} b_{\underline{1}j} + a_{i\underline{2}} b_{\underline{2}j} + \cdots + a_{i\underline{k}} b_{\underline{k}j}$$

Identity matrices are square matrices with 1's on the main diagonal and zeros everywhere else. They are denoted I or I_n if referencing the size, $n \times n$. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad AI_3 = A$

Theorem 1.4.3 If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has at least one row of zeros or R is the identity matrix I_n.

Pf: An $n \times n$ matrix has at most n pivots. If there are n pivots in R, then $R = I_n$ by def of rref. If there are fewer than n pivots, then at least one row doesn't have a leading 1 and so is a row of zeros.

Definition 1: If A is a square matrix, and if there exists a matrix B of the same size for which $AB = BA = I$, then A is said to be **invertible** (or **nonsingular**) and B is called the **inverse** of A. If no such matrix B exists, then A is said to be **singular**. Because the inverse of a matrix A is unique, we will denote it using A^{-1}.

Theorem 1.4.4 Uniqueness of a Matrix Inverse
If B and C are both inverses of the matrix A, then $B = C$.

Pf:

prop of I

$$B = BI = B(AC) = (AB)C = IC = C \checkmark$$

property of identity matrix

C is an inverse of A

matrix mult. is associative

B is an inverse of A

Theorem 1.4.5 The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

8. Use Theorem 1.4.5 to compute the inverse.

$\begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$

$$ad - bc = -6 - (-8) = 2$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & -4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -2 \\ 1 & 3 \end{bmatrix}$$

Theorem 1.4.6 If A and B are invertible matrices with the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

pf:

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

Since multiplying (AB) by $(B^{-1}A^{-1})$ yields I, $(AB)^{-1} = B^{-1}A^{-1}$.

Note: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

Theorem 1.4.7 Properties of Negative Exponents

If A is invertible and n is a nonnegative integer, then:

a) A^{-1} is invertible and $(A^{-1})^{-1} = A$.

b) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

reciprocal, inverse

c) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.

$\frac{1}{7}A^{-1}$

15. Use the given information to find A.

$(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$ $\Rightarrow$ $((7A)^{-1})^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}^{-1}$

$7A = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}^{-1}$

$7A = \frac{1}{-1}\begin{bmatrix} -2 & -7 \\ -1 & -3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2/7 & 1 \\ 1/7 & 3/7 \end{bmatrix}$

23. Find all values of a, b, c, and d (if any) for which the matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ commute.

A & B commute if $AB = BA$

$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$

$\boxed{\text{A \& B commute if } c=0,\ a=d, \text{ and } b \text{ is any number.}}$

$\frac{1}{3}\cancel{3}x = 15\,\frac{1}{3}$

$x = 5$

25. Solve the system using an inverse matrix.

$3x_1 - 2x_2 = -1$

$4x_1 + 5x_2 = 3$

$A\vec{x} = \vec{b} \Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{b}$

$\boxed{\vec{x} = A^{-1}\vec{b}}$

Matrix equation: $\begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & 5 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{15+8}\begin{bmatrix} 5 & 2 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5/23 & 2/23 \\ -4/23 & 3/23 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5/23 + 6/23 \\ 4/23 + 9/23 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/23 \\ 13/23 \end{bmatrix}}$$

39. Simplify the expression assuming that A, B, C and D are invertible.

$(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$

$B^{-1}A^{-1}AC^{-1}CDD^{-1} = B^{-1}$

Theorem 1.4.8 Properties of the Transpose

If the sizes of the matrices are such that the stated operations can be performed, then:

a) $(A^T)^T = A$

b) $(A+B)^T = A^T + B^T$

c) $(A-B)^T = A^T - B^T$

d) $(kA)^T = kA^T$

e) $\boxed{(AB)^T = B^TA^T}$

$m \times n \quad n \times k \qquad k \times n \quad n \times m$

Theorem 1.4.9 If A is an invertible matrix, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

A invertible $\Rightarrow A^{-1}$ exists.

$$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$$

Since $A^T(A^{-1})^T = I$, $(A^{-1})^T = (A^T)^{-1}$. ✓